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LETTER TO THE EDITOR

Reflections on the symmetry–conservation law duality and the Runge–Lenz vector

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Abstract. Following recent interest in an earlier work on the Kepler problem, a number of remarks are made concerning the many-faceted symmetry–conservation law duality in classical mechanics. In particular, the advantages of associating a number of symmetries with the Runge–Lenz vector are discussed.

A few years ago C J Eliezer and I produced some work on the classical Kepler problem (Prince and Eliezer 1981) which has recently raised interest in these pages and elsewhere (Schafir 1981, Sarlet and Cantrijn 1981a, b, Mariwalla 1982). In this letter I would like to make a number of remarks about the comments of Schafir and Mariwalla and to place our original results in a broader perspective.

The 'symmetry–conservation law' duality in physics has always been a popular and contentious issue. Perhaps the most outstanding feature of the idea is the multiplicity of its realisations, and herein lies the source of much of the dispute. The modern concept probably owes itself to Klein, whose statement in his Erlanger programme that geometries may be classified according to properties left invariant under groups of transformations was the precursor of his proposal for a study of the classification of conservation laws of differential equations according to the invariance group of the variational principle from which they are derived (Klein 1918a, b). Certainly Noether (1918) acknowledges his suggestion in the presentation of her famous theorem, although the considerations there are restricted to the invariance of the action integral rather than of the variational principle. The mathematical technique on which her work depended was Lie's theory of differential equations and continuous transformation groups, as have most subsequent approaches to the problem.

By the end of the last decade it had been established that the invariance group of the action integral was a subgroup of the invariance group of the variational principle as far as group action on configuration space is concerned (the so-called 'point' symmetries). Furthermore, while the 'Noether' symmetries led to closed form conservation laws, the larger class of 'Lie' symmetries in general did not (a notable exception being for special functional forms of the force giving the 'KLM' constants, see Mariwalla (1980) and references therein). However, this larger group has the advantage of not depending on the existence of a Lagrangian, only on the equation of motion, and there is an algorithm for producing associated first integrals using the solvable subalgebras of the associated Lie algebra (Prince (1981); this is the rigorous version of the considerations leading to the Runge–Lenz vector in our paper). Both types of symmetry have a clear geometric nature, being actions on the configuration space (M), and some geometric deformation of the trajectories is usually associated with the resulting conservation laws. (I refer the reader to the review paper of Sarlet and Cantrijn (1981b) for details of the development of Noether's theorem.)

In general, however, there will be conservation laws which are not associated in the above ways with symmetries, and of course mathematical physicists are wont to look for a one-to-one correspondence between such classes of related objects. At this point we need to have a clear idea of what we want to mean by the term invariance. No matter how much Lie's original calculations depended on invariance of functional forms (his work was very coordinate dependent), his underlying idea was that the action of a one-parameter invariance group should be to map trajectories into trajectories. It was this feature of his theory which attracted differential geometers and which led, through considerations of invariance of families of geodesics, to the theory of projective, affine, isometric and conformal invariance (and to the association of conservation laws and symmetries developed by Katzin and Levine and, independently, Mariwalla; see Eisenhart (1926, 1927), Yano (1955) and, for example, Katzin and Levine (1974), Mariwalla (1975)). Now, given this idea of invariance, Cartan had the simple relevation (in hindsight) that a one-parameter group acting on evolution space E ($\mathbb{R}\times TM$, $\mathbb{R}$ being for time) will transform the trajectories on M by lifting them to E, using the group, and then projecting the resulting curves back onto M. Importantly, such a transformation will usually not be the result of the action of a one-parameter group on $\mathbb{R}\times M$. By requiring that such a one-parameter group on E permute the trajectories (considered on E or M), Noether's theorem can be generalised, as can Lie's theory. What we get is a class of these 'dynamical' symmetries, called Cartan symmetries, which are in one-to-one correspondence with all the conservation laws of the system (by direct construction, see e.g. Crampin (1977)). The other features of this formulation are that all Noether symmetries are Cartan symmetries and all Lie symmetries are dynamical symmetries (see Prince 1983). However, the association of conservation law and symmetry may change if a non-trivially equivalent Lagrangian for the system is used—one reason not to be dogmatic about the nature of the duality (see Marmo and Saletan 1977).

The ruthless insistence on a one-to-one correspondence is at the expense of a good deal of geometric interpretation. For example, with the classical Kepler problem ($M=\mathbb{R}^2$) our vector field

$$Y_3 = t\partial/\partial t + \tfrac{2}{3}r\partial/\partial r \tag{1}$$

generates the finite transformations on $\mathbb{R}\times\mathbb{R}^2$

$$\bar{t} = t\exp(\alpha_3), \qquad \bar{r} = r\exp(\tfrac{2}{3}\alpha_3), \qquad \bar{\theta}=\theta \tag{2}$$

(α_3 being the group parameter), with *geometric* paths

$$r^3/t^2 = \text{constant}, \qquad \theta = \text{constant}. \tag{3}$$

In addition, Y_3 is the only Lie symmetry which non-trivially deforms the orbits leaving those geometric features associated with the Runge–Lenz vector, namely eccentricity and orbit orientation, invariant.

Using the Cartan approach, we find that the two vector fields on E associated with the two cartesian components of the Runge–Lenz vector are

$$Z_1 = x^1\frac{\partial}{\partial t} + (x^1\dot{x}^1 + x^2\dot{x}^2)\frac{\partial}{\partial x^1} + (x^2\dot{x}^1 - x^1\dot{x}^2)\frac{\partial}{\partial x^2} + \left(\dot{x}^{22} - \frac{\mu}{r}\right)\frac{\partial}{\partial \dot{x}^1} - \dot{x}^1\dot{x}^2\frac{\partial}{\partial \dot{x}^2} \tag{4}$$

and

$$Z_2 = x^2 \frac{\partial}{\partial t} + (x^1\dot{x}^2 - x^2\dot{x}^1)\frac{\partial}{\partial x^1} + (x^1\dot{x}^1 + x^2\dot{x}^2)\frac{\partial}{\partial x^2} - \dot{x}^1\dot{x}^2\frac{\partial}{\partial \dot{x}^1} + \left(\dot{x}^{12} - \frac{\mu}{r}\right)\frac{\partial}{\partial \dot{x}^2}, \tag{5}$$

up to a multiple of the fundamental vector field

$$\Gamma = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} - \frac{\mu x^i}{r^3}\frac{\partial}{\partial \dot{x}^i} \tag{6}$$

(Prince 1981, Sarlet and Cantrijn 1981a, b). This correspondence is one-to-one but not laden with geometric import as far as M or $\mathbb{R}\times M$ is concerned.

So, when it is insisted that *only* the energy can be associated with the dilatation symmetry (1) (Mariwalla 1982), it must be understood which association between conservation laws and symmetries is being used. Also, when a one-to-one correspondence is insisted upon (Schafir 1981), we must realise that it will not give us a great deal of insight into the geometrical invariances of the orbits, even though it may be necessary when no Lie symmetries exist to 'correspond' to a given conservation law. The overriding lesson is that the symmetry–conservation law duality is more valuable considered in all its manifestations rather than in any particular one of them.

Schafir also raises the question of whether our integration procedure gives any regard to the vectorial nature of the Runge–Lenz vector. In particular, he raises the objection that the 'transformed infinitesimal invariance is not associated with the transformed (as a vector) constant' (my paraphrasing). The following observations may clarify the point. Consider the Kepler problem in two dimensions again and use cartesian coordinates (x^1, x^2). Then every point on M has an orbit through it and hence a Runge–Lenz vector attached to it. The vector field thus defined is

$$\boldsymbol{R} = R^i \partial/\partial x^i = [x^i(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}) - \dot{x}^i(\boldsymbol{r}\cdot\dot{\boldsymbol{r}}) - \mu x^i/r]\partial/\partial x^i, \tag{7}$$

where the $\dot{x}^i$ are understood to be given by the tangent vector field to the orbits at each point (x^1, x^2). Now, on either $\mathbb{R}\times M$ or $\mathbb{R}\times TM$ (the prolongation of $\boldsymbol{R}$ to $\mathbb{R}\times TM$ is just $\boldsymbol{R}^{(1)} = \boldsymbol{R}$ because $\Gamma(R^i) = 0$),

$$[Y_3, \boldsymbol{R}] = [Y_3^{(1)}, \boldsymbol{R}^{(1)}] = -\tfrac{2}{3}\boldsymbol{R}, \tag{8}$$

that is,

$$\mathscr{L}_{Y_3^{(1)}}\boldsymbol{R}^{(1)} = -\tfrac{2}{3}\boldsymbol{R}^{(1)}, \tag{9}$$

which simply says that $\boldsymbol{R}$ is transformed (as a vector) into itself under the flow of Y_3. Furthermore, in as much as the vector fields Z_1, Z_2 'represent' R^1, R^2, the salient transformation properties are, not surprisingly,

$$[Y_3^{(1)}, Z_a] = \tfrac{1}{3}Z_a \tag{10}$$

(see Prince (1983): in the application of the rigorous integration procedure found in Prince (1981) I have shown that Y_3 leads to the scalar constant $\|\boldsymbol{R}\|$, and moreover $[Y_3, \partial/\partial\theta] = 0$).

Finally, I want to say something positive about the Cartan approach to the symmetry–conservation law duality. In differential geometry, particularly as it is used in general relativity, the technique allows a very thoroughgoing formulation and generalisation of the 'projective differential geometry' of Eisenhart and others. Specifically, it allows (i) the previously troublesome induced proper time variation to be simply dealt with, (ii) the appearance of proper time dependent conserved quantities and (iii) a simple relation between Killing tensors and vector fields on the evolution space ($\mathbb{R} \times TM$; M being the usual space-time). These results will be the subject of a future paper.

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